# A Note on Rational Approximation to $(1-x)^\sigma$

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THEOREM 1.   *For $0 < \sigma \leqslant 1$, $0 \leqslant x \leqslant 1$ and $n \geqslant 0$, we have*

$$0 \leqslant S_n^{-1}(x) - (1-x)^\sigma \leqslant \frac{x^{n+1}}{S_n(x)} \leqslant \binom{\sigma + n}{n}^{-1}, \tag{1}$$

*where $S_n(x) = \sum_{j=0}^{n} \binom{\sigma + j - 1}{j} x^j$. (Note that $(1-x)^{-\sigma} = \sum_{j=0}^{\infty} \binom{\sigma + j - 1}{j} x^j$ for $0 < \sigma \leqslant 1$, $0 \leqslant x < 1$.)*

*Proof.*   For $0 < \sigma \leqslant 1$, $0 \leqslant x < 1$, $j \geqslant 0$, and $n \geqslant 0$, we obtain, by oberserving the fact that

$$\binom{n + \sigma + j}{n + 1 + j} \leqslant \binom{\sigma + j - 1}{j},$$

$$0 \leqslant S_n^{-1}(x) - (1-x)^\sigma = \frac{\sum_{j=n+1}^{\infty} \binom{\sigma + j - 1}{j} x^j}{S_n(x)(1-x)^{-\sigma}}$$

$$= \frac{x^{n+1} \sum_{j=0}^{\infty} \binom{n + \sigma + j}{n + 1 + j} x^j}{S_n(x)(1-x)^{-\sigma}}$$

$$\leqslant \frac{x^{n+1} \sum_{j=0}^{\infty} \binom{\sigma + j - 1}{j} x^j}{S_n(x)(1-x)^{-\sigma}} = \frac{x^{n+1}}{S_n(x)}.$$

Let $0 < \sigma \leqslant 1$, $n \geqslant 0$. As

$$(n + 1) S_n(x) > x S_n'(x) \qquad \text{for} \quad x > 0,$$

$$\left( \frac{x^{n+1}}{S_n(x)} \right)' > 0 \qquad \text{for} \quad x > 0,$$

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and hence $x^{n+1}/S_n(x)$ is increasing on $[0, \infty)$. Thus, for $0 \leqslant x < 1$,

$$0 \leqslant [S_n(x)]^{-1} - (1-x)^\sigma \leqslant \frac{x^{n+1}}{S_n(x)} < \frac{1}{S_n(1)}$$

$$= \left( \sum_{j=0}^{n} \binom{\sigma+j-1}{j} \right)^{-1} = \binom{\sigma+n}{n}^{-1}.$$

The last two equalities prove (1) also for $x = 1$. It follows that

$$\| (1-x)^\sigma - S_n^{-1}(x) \|_{L^\infty[0,1]} \leqslant \binom{\sigma+n}{n}^{-1}. \tag{2}$$

*Remarks on Theorem* 1.    (i) For $\sigma = \frac{1}{2}$, (2) was obtained in [1].

(ii)   For $0 < \sigma < 1$, (2) is sharper than the estimate given in [2], as for $n = 1, 2, ...,$

$$\binom{\sigma+n}{n}^{-1} = \frac{\Gamma(\sigma+1)\,\Gamma(n+1)}{\Gamma(\sigma+n+1)} < \frac{n}{n+1} \frac{\Gamma(\sigma)\,\Gamma(n)}{\Gamma(\sigma+n)}.$$

THEOREM 2.   *Let $P(x)$ and $Q(x)$ be polynomials of degree at most $n$ $(\geqslant 1)$ having only real, non-negative coefficients $(Q(0) > 0)$ and let $0 < \sigma < 1$. Then*

$$\left\| (1-x)^\sigma - \frac{P(x)}{Q(x)} \right\|_{L^\infty[0,1]}$$

$$\geqslant (n+1)^{-\sigma} \left( 1 + \left( \frac{n+1}{n} \right)^n \right)^{-1} > \frac{(n+1)^{-\sigma}}{1+e}. \tag{3}$$

*Proof.*   Set

$$\left\| (1-x)^\sigma - \frac{P(x)}{Q(x)} \right\|_{L^\infty[0,1]} = \varepsilon.$$

Then

$$\varepsilon \geqslant \left( 1 - \frac{n}{n+1} \right)^\sigma - \frac{P\left( \dfrac{n}{n+1} \right)}{Q\left( \dfrac{n}{n+1} \right)}$$

$$\geqslant \frac{1}{(n+1)^\sigma} + \left( 0 - \frac{P(1)}{Q(1)} \right) \left( \frac{n+1}{n} \right)^n$$

$$\geqslant \frac{1}{(n+1)^\sigma} - \varepsilon \left( \frac{n+1}{n} \right)^n$$

Hence

$$\varepsilon \geqslant (n+1)^{-\sigma} \left( 1 + \left( \frac{n+1}{n} \right)^n \right)^{-1}.$$

*Remarks on Theorem* 2. (i) The estimate (3) is sharper than these obtained in [2, 3]. The technique used here is different from the one used in [3].

(ii) Actually the inequality $P(1) \geqslant \max(0, P(n/(n+1)))$ suffices for the proof of Theorem 2 rather than assuming $P(x)$ has only real, non-negative coefficients.

THEOREM 3. *Let $P(x)$ and $Q(x)$ be polynomials of degree at most $n$ ($\geqslant 2$) and let $P(x)$ have only real, non-negative coefficients. Let $Q(x) \neq 0$ throughout $[0, 1]$ and let $0 < \sigma \leqslant 1$. Then*

$$\left\| (1-x)^\sigma - \frac{P(x)}{Q(x)} \right\|_{L^\infty[0,1]} \geqslant (16n^{2\sigma})^{-1}.$$

We need the following well known

LEMMA [5, p. 68, Eq. (9)]. *If a real polynomial $Q_n(x)$ of degree $\leqslant n$ ($n \geqslant 0$) satisfies the inequality $|Q_n(x)| \leqslant L$ on $[a, b]$ ($-\infty < a < b < \infty$), then at each real $x$ outside $[a, b]$ we have*

$$|Q_n(x)| \leqslant L \left| T_n \left( \frac{2x - a - b}{b - a} \right) \right|,$$

*where $T_n$ is the $n$th degree Chebychev polynomial of the first kind.*

*Proof of Theorem* 3. Suppose

$$\left\| (1-x)^\sigma - \frac{P(x)}{Q(x)} \right\|_{L^\infty[0,1]} < (16n^{2\sigma})^{-1}. \tag{4}$$

Then on $[0, 1 - n^{-2}]$,

$$\left| \frac{P(x)}{Q(x)} \right| > (1-x)^\sigma - \frac{1}{16n^{2\sigma}} \geqslant \frac{1}{n^{2\sigma}} - \frac{1}{16n^{2\sigma}} = \frac{15}{16n^{2\sigma}},$$

$$|Q(x)| < \frac{16n^{2\sigma}}{15} P(1). \tag{5}$$

In (5) we have used the assumption that $P(x)$ has only real, non-negative coefficients. Now, by applying the lemma to (5) with $a = 0$, $b = 1 - n^{-2}$, we

get on $[0, 1]$, by familiar elementary properties of $T_n$, and by [4, p. 38, Problem 172],

$$
\begin{aligned}
\underset{[0,1]}{\text{Max}} |Q(x)| &\leqslant \frac{16n^{2\sigma}}{15} P(1) \, T_n \left( \frac{n^2+1}{n^2-1} \right) \\
&< \frac{16n^{2\sigma}}{15} P(1) \left( \frac{n^2+1}{n^2-1} + \sqrt{\left( \frac{n^2+1}{n^2-1} \right)^2 - 1} \right)^n \\
&= \frac{16n^{2\sigma}}{15} P(1) \left( \frac{n+1}{n-1} \right)^n \\
&\leqslant \frac{16n^{2\sigma}}{15} P(1) \cdot 9 = \frac{48}{5} n^{2\sigma} P(1).
\end{aligned}
\tag{6}
$$

From (6) and (4) we obtain $\frac{5}{48} < n^{2\sigma} |P(1)/Q(1)| < \frac{1}{16}$, which is false.

## REFERENCES

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